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INFRARED RADIATIVE HEAT TRANSFER IN NONGRAY GASES*

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Abstract—The object of this investigation is to study various approximate methods of analyzing infrared radiative heat transfer in nongray nonisothermal gases. For this purpose, a very simple physical system was chosen consisting of a gas bounded by two infinite parallel black plates having the same uniform temperature. There is a uniform heat source (or sink) within the gas. Furthermore, attention is restricted to gases having a single fundamental vibration–rotation band; that is, diatomic gases.

It is found that for intermediate optical thicknesses, the line structure of the vibration–rotation band can have a significant effect upon the temperature distribution within the gas. Predictions based on the gray gas assumption are shown to be greatly in error. It is further illustrated that one cannot apply the optically thick limit to an entire vibration–rotation band, since there will always be an optically non-thick region in the band wings, and such regions will contribute significantly to the radiative transfer process.

NOMENCLATURE

A, total band absorptance [cm^{-1}];
A_0, band width parameter [cm^{-1}];
B^2, line width parameter [atm^{-1}];
C_0^2, correlation parameter [$atm^{-1}\,cm^{-1}$];
e, total black body emissive power [W/cm^2];
e_ω, Plank's function [$W/cm^2/cm^{-1}$];
e_{1_ω}, Plank's function evaluated at temperature T_1;
e_{ω_c}, Planck's function evaluated at wave number ω_c;
$E_n(t)$, exponential integral;
L, distance between plates [cm];
P, pressure [atm];
q_R, total radiation heat flux [W/cm^2];
$q_{R\omega}$, spectral radiation heat flux [$W/cm^2/cm^{-1}$];
Q, heat source or sink [W/cm^3];
$S(T)$, total band intensity [$atm^{-1}\,cm^{-2}$];
T, temperature [°K];
T_1, wall temperature;
u, dimensionless coordinate, C_0^2Py;
u_0, dimensionless path length, C_0^2PL;
w, pressure path length, Py [atm cm];
y, physical coordinate [cm].

Greek symbols

β, line structure parameter, $B^2P_e = 1{\cdot}02\,B^2P$;
$\Delta\omega$, effective band width [cm^{-1}];
κ_ω, spectral absorption coefficient [cm^{-1}];
κ_m, modified Planck mean coefficient [cm^{-1}];
κ_P, Plank mean coefficient [cm^{-1}];
κ_R, Rosseland mean coefficient [cm^{-1}];
σ, Stefan–Boltzmann constant;
τ_ω, optical coordinate, $\kappa_\omega y$;
$\tau_{0\omega}$, optical thickness, $\kappa_\omega L$;
τ, optical coordinate, $\kappa_P y$;
τ_0, optical thickness, $\kappa_P L$;
$\bar{\tau}$, optical coordinate, $\bar{\kappa} y$;
$\bar{\tau}_0$, optical thickness, $\bar{\kappa} L$;
φ, dimensionless function defined by equation (28);

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ω, wave number [cm^{-1}];
ω_c, band center [cm^{-1}].

INTRODUCTION

ANALYSES of radiative heat transfer within absorbing–emitting media have recently received considerable attention. With but few exceptions, however, this work has been restricted to the assumption of a gray medium. Moreover, virtually no comparisons have been made between the analyses which utilize various nongray models.

The purpose of the present paper is to investigate and compare several methods of analyzing nongray radiative transfer in gases. Attention will be directed specifically towards infrared radiation in diatomic gases, wherein the absorption and emission of thermal radiation occurs as the result of vibration–rotation bands. Diatomic gases will radiate in the infrared only if they possess unsymmetric molecules, with the single fundamental band being of main importance.

Existing work in this area has been concerned with specific problems and has dealt primarily with the more complicated case of polyatomic gases. Combined conduction and radiation in ammonia has been studied both experimentally and analytically by Gille and Goody [1]. It was illustrated that the temperature profile within the gas could be described by an integro-differential equation for which the kernel was the first derivative of the total gas emittance. This kernel function was evaluated for ammonia through use of the statistical model for vibration–rotation bands. Another experimental and analytical study is that of Nichols [2], which deals with turbulent flow of water vapor in the entrance region of an annular duct. The statistical model was again employed, although the procedure differed somewhat from that of Gille and Goody [1]. Laminar flow of carbon dioxide in the entrance region of a circular tube has been analyzed by de Soto and Edwards [3]. In this case an exponential model was used to describe the spectral absorption coefficient for each vibration–rotation band.

In addition to these specific nongray solutions, Wang [4] has presented a general formulation for the case of a gas having a single vibration–rotation band (diatomic gas). It was shown that the one-dimensional divergence of the radiation flux vector can be described in terms of spatial integrals involving the first and second derivatives of the total band absorptance.

In the present investigation, comparative solutions will be obtained for the case of a gas bounded between two parallel black plates and within which there is a uniform heat source. For comparisons in which it is necessary to specify a particular gas, carbon monoxide will be utilized. It should again be emphasized that the sole purpose of the present paper is the investigation of analytical methods for treating radiative transfer in nongray gases. Consequently, the physically unreal assumption of negligible thermal conduction within the gas will be employed. It will additionally be assumed that the gas is in local thermodynamic equilibrium, i.e. that the populations of vibrational and rotational energy states are collision controlled.

BASIC EQUATIONS

The physical model and coordinate system are illustrated in Fig. 1. A gas containing a uniform heat source (or sink) per unit volume, Q, is bounded by two infinite parallel black plates having the same uniform temperature T_1.

It will be assumed that the spectral absorption coefficient of the gas, κ_ω, is independent of temperature, i.e. restriction is made to moderately small temperature differences within the gas. With this assumption, the optical coordinate and thickness are, on a spectral basis, respectively

$$\tau_\omega = \kappa_\omega y \qquad \tau_{0\omega} = \kappa_\omega L.$$

Furthermore, from [5] the spectral radiation

heat flux within the medium may be expressed as

$$q_{R\omega} = 2\int_0^{\tau_\omega} [e_\omega(t) - e_{1\omega}]\, E_2(\tau_\omega - t)\, dt - 2\int_{\tau_\omega}^{\tau_{0\omega}} [e_\omega(t) - e_{1\omega}]\, E_2(t - \tau_\omega)\, dt. \quad (1)$$

Since it is assumed that the only other mechanism of energy transfer to or from the

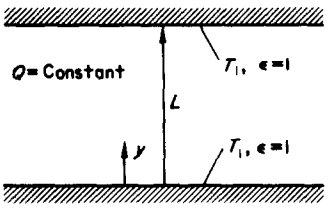

FIG. 1. Physical model and coordinate system.

gas is the uniform heat source Q, then from conservation of energy

$$\frac{dq_R}{dy} = Q \quad (2)$$

where q_R is the total radiative flux over all values of wave number; i.e.

$$q_R = \int_0^\infty q_{R\omega}\, d\omega. \quad (3)$$

From symmetry it follows that $q_R = 0$ at $y = L/2$, and equation (2) may be integrated to yield

$$q_R = \frac{QL}{2}\left(2\frac{y}{L} - 1\right). \quad (4)$$

Equation (4), when combined with equations (1) and (3), yields the integral equation describing the temperature profile within the gas.

An often employed simplification to equation (1) is obtained through use of the exponential kernel approximation [5]

$$E_2(t) \simeq \tfrac{3}{4} e^{-3t/2} \quad (5)$$

and this will be employed in the present work. For the gray medium, approximation to the present problem by use of the constants appearing in equation (5) gives the best overall agreement with the exact solution of Heaslet and Warming [6].

BAND INFORMATION

Diatomic absorbing–emitting gases, such as carbon monoxide, have a single fundamental band as well as overtone bands. Usually, however, overtone bands may be neglected, and only the fundamental band will be included in the following development. The first overtone band of carbon monoxide, for example, has roughly one per cent of the intensity of the fundamental band.

The line-averaged absorption coefficient for the fundamental band of carbon monoxide is illustrated in Fig. 2. However, this constitutes a usable result for the variation of κ_ω with wave

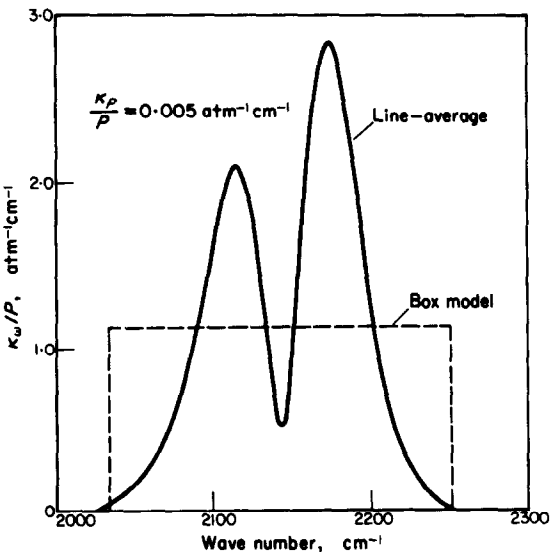

FIG. 2. Spectral absorption coefficient of carbon monoxide at room temperature.

number only in the sense of a high-pressure limit, i.e. when the individual rotational lines of the band are sufficiently pressure broadened so as to produce a uniform variation of κ_ω

with wave number. The area under the κ_ω/P vs. wave number curve is the band intensity

$$S(T) = \int_{\Delta\omega} \frac{\kappa_\omega}{P} \mathrm{d}\omega \tag{6}$$

and for the CO fundamental band is evaluated as [7]

$$S(T) = 237 \left(\frac{300}{T}\right). \tag{7}$$

As illustrated in [5], optically thin radiation can often be formulated in terms of two mean coefficients. The first is the Planck mean coefficient, which is actually a mean *emission* coefficient, defined as

$$\kappa_P(T) = \frac{\int_{\Delta\omega} \kappa_\omega(T) e_\omega(T) \mathrm{d}\omega}{e(T)}. \tag{8}$$

Since the band width, $\Delta\omega$, is quite small, $e_\omega(T)$ may be assumed to be independent of wave number within the band, and letting $e_{\omega_c}(T)$ denote $e_\omega(T)$ evaluated at the band center, it follows from equation (6) that

$$\frac{\kappa_P}{P} = \frac{e_{\omega_c}(T)}{\sigma T^4} S(T). \tag{9}$$

The center of the CO fundamental band is located at [8]:

$$\omega_c = 2143\ \mathrm{cm}^{-1}. \tag{10}$$

Equation (9) is precisely the result given by Abu-Romia and Tien [9], and, as discussed by them, illustrates that κ_P/P is independent of the actual line structure of the band (which of course is the reason for the invariance with pressure).

The second absorption coefficient applicable to the optically thin limit is the modified Planck mean coefficient

$$\kappa_m(T, T_1) = \frac{\int_{\Delta\omega} \kappa_\omega(T) e_\omega(T_1) \mathrm{d}\omega}{e(T_1)}. \tag{11}$$

Following the same procedure used in arriving at equation (9) and noting that $S(T) \sim 1/T$ for a fundamental band

$$\frac{\kappa_m(T, T_1)}{P} = \frac{e_{\omega_c}(T_1)}{\sigma T_1^4} S(T) = \frac{\kappa_P(T_1)}{P} \frac{T_1}{T}. \tag{12}$$

Although neither $\kappa_P(T)$ nor $\kappa_m(T, T_1)$ are dependent upon the line structure of the band, the line structure will influence the range of applicability of the optically thin limit. This is due to the fact that the gas must be optically thin for *all* values of wave number, such that $(\kappa_\omega)_{\max} L \ll 1$, where $(\kappa_\omega)_{\max}$ denotes the maximum value of κ_ω within the band. This maximum value of the absorption coefficient will correspond to the peak of the strongest rotational lines within the band, and this may considerably exceed the maximum value for the line-averaged absorption coefficient as would be obtained, for example, from Fig. 2.

A quantity which will prove useful in the following section dealing with nongray solutions is the total band absorptance, which is defined as

$$A = \int_{\Delta\omega} \left[1 - \exp\left(-\frac{\kappa_\omega}{P} w\right)\right] \mathrm{d}\omega \tag{13}$$

where $w = Py$. With the exception of very high pressures, the evaluation of equation (13) would involve integration over the discrete line structure of the band, and this would prove to be a formidable task. However, through the use of simplified band models, considerable information can be deduced concerning the total band absorptance. For example, employing the model of a vibrating nonrigid rotator, Edwards and Menard [10] have shown that the total band absorptance possesses a logarithmic asymptote for large values of the path length w. This asymptotic behavior had previously been observed experimentally [11]. Furthermore, the logarithmic asymptote applies when the path length is sufficiently large such that the central portion of the band is opaque, and radiation transfer within the gas takes place solely in the wing regions of the band.

Further conditions on the total band absorptance have been given by Tien and Lowder [12],

from which they arrived at the correlation

$$A = A_0 \ln\left\{uf(\beta)\left[\frac{u+2}{u+2f(\beta)}\right]+1\right\} \quad (14)$$

where

$$u = C_0^2 Py \qquad \beta = B^2 P_e$$

$$f(\beta) = 2{\cdot}94[1-\exp(-2{\cdot}60\,\beta)]$$

and

$$A_0 C_0^2 = S(T). \quad (15)$$

In addition, for the CO fundamental band [13]

$$A_0 = 38\left(\frac{T}{300}\right)^{\frac{1}{2}} \quad (16a)$$

$$B^2 = 0{\cdot}0838\left(\frac{300}{T}\right)^{0{\cdot}42} \quad (16b)$$

$$P_e = 1{\cdot}02P. \quad (16c)$$

It is important to recognize that equation (14) does, at least in a semi-empirical sense, account for the line structure of the band. This line structure dependency introduces the dimensionless pressure, β, as a parameter.

A very simple band approximation, which has seen application in the calculation of gas emittances [8], is the box model. This is illustrated in Fig. 2, and it is assumed that κ_ω is constant over an effective band width $\Delta\omega$. It is natural to require that the area under the approximated band is conserved; that is

$$\bar{\kappa}\Delta\omega = PS(T). \quad (17)$$

From Penner [8], the effective band width for the CO fundamental band may be expressed by

$$\Delta\omega = 214\left(\frac{T}{300}\right)^{\frac{1}{2}}. \quad (18)$$

Thus, for the CO fundamental, equations (7, 17, 18) yield

$$\frac{\bar{\kappa}}{P} = 1{\cdot}11\left(\frac{300}{T}\right)^{\frac{1}{2}}. \quad (19)$$

The box model has the same failings in the present application as it does in the calculation of gas emittances [8]. Since the model in no way accounts for the line structure of the band, it is restricted to moderately high pressures. It would further be expected that the box model should fail for large path lengths, since it does not account for the wing regions of the band, but arbitrarily cuts off the band at $\omega_c \pm \Delta\omega/2$.

RADIATIVE TRANSFER ANALYSES

In this section several approximate methods of solving the present problem will be illustrated. With the exception of the optically thin limit, the solutions are consistent with the application of the exponential kernel approximation as described by equation (5). The various solutions pertain to different methods of approximating the spectral behavior of the absorption coefficient κ_ω. As previously discussed, it will be assumed that the spectral absorption coefficient is independent of temperature, and κ_ω will be evaluated at the temperature T_1.

Gray gas

The gray gas assumption replaces the wave-number dependent absorption coefficient by a wave-number averaged quantity. For lack of a more rational choice, this average coefficient will be taken to be $\kappa_P(T_1)$, and thus

$$\tau = \kappa_P y \qquad \tau_0 = \kappa_P L.$$

On combining equations (1, 3, 4, 5) the integral equation describing the gray gas problem becomes

$$\frac{Q\tau_0}{2\sigma\kappa_P}\left(2\frac{\tau}{\tau_0}-1\right) = \frac{3}{2}\int_0^\tau [T^4(t)-T_1^4]\exp[-\tfrac{3}{2}(\tau-t)]\,dt - \frac{3}{2}\int_\tau^{\tau_0} [T^4(t)-T_1^4]\exp[-\tfrac{3}{2}(t-\tau)]\,dt. \quad (20)$$

When this equation is differentiated twice, the integrals repeat themselves and may be

eliminated, and the solution for the temperature profile within the gas is

$$\frac{T^4 - T_1^4}{Q/\sigma\kappa_P} = \frac{1}{3} + \frac{\tau_0}{4} + \tfrac{3}{8}\tau_0^2\left(\frac{\tau}{\tau_0} - \frac{\tau^2}{\tau_0^2}\right). \quad (21)$$

Note that under optically thin conditions ($\tau_0 \to 0$)

$$\frac{T^4 - T_1^4}{Q/\sigma\kappa_P} = \frac{1}{3}. \quad (22)$$

Optically thin limit

An exact formulation of the nongray problem is possible in the optically thin limit. From [5]

$$-\frac{dq_R}{dy} = 4\sigma\kappa_m(T, T_1)T_1^4 - 4\sigma\kappa_P(T)T^4. \quad (23)$$

Combining this with equation (2), and making use of the result

$$\kappa_m(T, T_1) = \kappa_P(T_1)\frac{T_1}{T} \quad (24)$$

as given by equation (12), there is obtained

$$\kappa_P(T)T^4 - \kappa_P(T_1)\frac{T_1^5}{T} = \frac{Q}{4\sigma}. \quad (25)$$

Since later solutions will describe the gas temperature in terms of the Planck function evaluated at the band center, it will be convenient to rephrase equation (25), through use of equation (9), and since $S(T) \sim 1/T$, to yield

$$e_{\omega_c}(T) - e_{\omega_c}(T_1) = \frac{Q}{4PS(T)}.$$

Since $\kappa_\omega(T)$ is assumed to be independent of temperature and evaluated at the temperature T_1, $S(T)$ may be replaced by $S(T_1)$ in the above expression, with the result

$$\frac{e_{\omega_c}(T) - e_{\omega_c}(T_1)}{Q/PS(T_1)} = \frac{1}{4}. \quad (26)$$

Equation (26) could also be obtained by first allowing the spectral absorption coefficient to be temperature dependent, and then going to the limit of small temperature differences. This would not be the case, however, if one employs equation (25) and then assumes that the Planck mean absorption coefficient is independent of temperature. In other words, the correct formulation for small temperature differences corresponds to assuming that the spectral coefficient, and *not* the Planck mean, is independent of temperature.

It is of interest to compare equation (26) with equation (22), which is the optically thin form of the gray gas solution. There are two distinct differences in these results. The first involves the numerical coefficient appearing on the right side of each of the equations ($\frac{1}{3}$ and $\frac{1}{4}$), and this difference has nothing to do with the spectral formulation of the problem. The value of $\frac{1}{3}$ appearing in equation (22) is a consequence of the exponential kernel approximation, which, as given by equation (5), produces the greatest error for optically thin conditions.

Aside from this difference in a numerical constant, equations (22) and (26) give, as will be shown later, decidedly different results for the gas temperature. The reason for this is that the gray gas solution is incorrect even in the optically thin limit. This is due to the fact that the gray gas solution is formulated solely in terms of the Planck mean coefficient, whereas the correct optically thin formulation involves both the Planck mean and modified Planck mean coefficients. Furthermore, it has been assumed in the gray gas analysis that the Planck mean coefficient is independent of temperature, and from previous arguments this is incorrect even for small temperature differences.

Box model

Consider now the application of the box model. Letting

$$\bar{\tau} = \bar{\kappa}y \qquad \bar{\tau}_0 = \bar{\kappa}L$$

where $\bar{\kappa}$ is defined by equation (17), and recalling that the absorption coefficient is assumed to be zero outside of the effective band width $\Delta\omega$,

equations (1, 3, 4, 5) combine to yield

$$\frac{\bar{\tau}_0}{2}\left(2\frac{\bar{\tau}}{\tau_0}-1\right)=\frac{3}{2}\int_0^{\bar{\tau}}\varphi(t)\exp\left[-\tfrac{3}{2}(\bar{\tau}-t)\right]dt - \frac{3}{2}\int_{\bar{\tau}}^{\bar{\tau}_0}\varphi(t)\exp\left[-\tfrac{3}{2}(t-\bar{\tau})\right]dt \qquad (27)$$

where

$$\varphi=\frac{e_{\omega_c}(T)-e_{\omega_c}(T_1)}{Q/\bar{\kappa}\Delta\omega}. \qquad (28)$$

Equation (27) is of the same form as the gray gas equation (20), and the temperature profile, described in terms of Planck's function evaluated at the band center, is found to be

$$\varphi(\bar{\tau})=\frac{1}{3}+\frac{\bar{\tau}_0}{4}+\tfrac{3}{8}\bar{\tau}_0^2\left(\frac{\bar{\tau}}{\bar{\tau}_0}-\frac{\bar{\tau}^2}{\bar{\tau}_0^2}\right). \qquad (29)$$

Note that for optically thin conditions ($\bar{\tau}\to 0$), equation (29) reduces to

$$e_{\omega_c}(T)-e_{\omega_c}(T_1)=\frac{1}{3}\frac{Q}{\bar{\kappa}\Delta\omega}=\frac{1}{3}\frac{Q}{PS(T_1)} \qquad (30)$$

With the exception of the previously discussed numerical factor, equation (30) is identical to the optically thin result as given by equation (26); that is, the box model yields the correct optically thin limit. This should be expected, since the formulation of the optically thin limit is independent of the actual variation of κ_ω with wave number, but depends only upon the area under the κ_ω/P vs. ω curve. Recall that the box model satisfies the correct area requirement as given by equation (17).

Band absorptance model

A solution will now be obtained which accounts in an approximate manner for both the line structure and the wings of the band. This is accomplished by expressing the kernel for the integral equation in terms of the total band absorptance given by equation (14). The method of formulating the integral equation is somewhat analogous to that employed by Gille and Goody [1], while it constitutes an approximate kernel application of Wang's formulation [4].

It will first be convenient to recast equation (1) in terms of the physical coordinate y, that is

$$y=\frac{\tau_\omega}{\kappa_\omega},\qquad L=\frac{\tau_{0\omega}}{\kappa_\omega},\qquad z=\frac{t}{\kappa_\omega}$$

so that equations (1) and (5) yield

$$q_{R\omega}=\frac{3}{2}\int_0^y[e_\omega(z)-e_{1\omega}]\kappa_\omega \exp\left[-\frac{3\kappa_\omega}{2}(y-z)\right]dz - \frac{3}{2}\int_y^L[e_\omega(z)-e_{1\omega}]\kappa_\omega \exp\left[-\frac{3\kappa_\omega}{2}(z-y)\right]dz. \qquad (31)$$

Again assuming that Planck's function is independent of wave number within the band, and noting from equation (13) that

$$A'(y)=\int_{\Delta\omega}\kappa_\omega e^{-\kappa_\omega y}\,d\omega$$

then equations (3) and (31) yield

$$q_R=\tfrac{3}{2}\int_0^y[e_{\omega_c}(z)-e_{1\omega_c}]\,A'[\tfrac{3}{2}(y-z)]\,dz - \tfrac{3}{2}\int_y^L[e_{\omega_c}(z)-e_{1\omega_c}]\,A'[\tfrac{3}{2}(z-y)]\,dz. \qquad (32)$$

In that the correlation for the total band absorptance is in terms of the dimensionless path length u, it will be convenient to convert equation (32) from y to u. Note first, however, that from equation (15)

$$u=C_0^2Py=\frac{SP}{A_0}y$$

while from equations (16a, 17, 18)

$$u=\frac{214}{38}\frac{SP}{\Delta\omega}y=\frac{214}{38}\bar{\kappa}y=\frac{214}{38}\bar{\tau} \qquad (33)$$

and thus u is an optical coordinate directly

related to that employed in the box model. With $u_0 = C_0^2 PL$, and letting u' be the dummy variable for u, equation (33) becomes

$$q_R = \tfrac{3}{2}\int_0^u [e_{\omega_c}(u') - e_{1\omega_c}]\, A'[\tfrac{3}{2}(u-u')]\, du' - \tfrac{3}{2}\int_u^{u_0} [e_{\omega_c}(u') - e_{1\omega_c}]\, A'[\tfrac{3}{2}(u'-u)]\, du' \quad (34)$$

where $A'(u)$ denotes the derivative of $A(u)$ with respect to u.

The dimensionless quantity φ, which was introduced in the box model solution, will again be employed, and note from equations (15, 17, 28) that

$$\varphi = \frac{e_{\omega_c}(T) - e_{1\omega_c}}{Q/\bar{\kappa}\,\Delta\omega} = \frac{e_{\omega_c}(T) - e_{1\omega_c}}{Q/PS} = \frac{e_{\omega_c}(T) - e_{1\omega_c}}{Q/A_0 C_0^2 P}. \quad (35)$$

Furthermore, the dimensionless band absorptance $\bar{A}$ is defined by $\bar{A} = A/A_0$, and upon combining equations (4) and (34), the integral equation describing the present problem becomes

$$\frac{u_0}{2}\left(2\frac{u}{u_0} - 1\right) = \frac{3}{2}\int_0^u \varphi(u')\, \bar{A}'[\tfrac{3}{2}(u-u')]\, du' - \frac{3}{2}\int_u^{u_0} \varphi(u')\, \bar{A}'[\tfrac{3}{2}(u'-u)]\, du'. \quad (36)$$

The solution of equation (36) has been accomplished by the method of undetermined parameters. A polynomial solution for φ was assumed, and the constants evaluated by satisfying the integral equation at equally spaced locations. Both quadratic and quartic solutions were utilized, with the two solutions yielding virtually identical results.

Optically thick limit

From Abu-Romia and Tien [9], the radiative flux for optically thick radiation is expressed as

$$q_R = -\frac{4\sigma}{3\kappa_R}\frac{dT^4}{dy} \quad (37)$$

where κ_R is a Rosseland coefficient associated with the fundamental band and is defined by

$$\frac{1}{\kappa_R} = \int_{\Delta\omega} \frac{1}{\kappa_\omega}\frac{de_\omega}{de}\, d\omega. \quad (38)$$

The flux q_R as described by equation (37) is actually the flux within the wave number region of the band. Due to the symmetry of the present problem, however, there is no net flux outside the band.

On combining equations (4) and (37), integrating, and making use of the fact that temperature continuity at the wall is achieved under optically thick conditions, one has

$$T^4 - T_1^4 = \frac{3\kappa_R Q L^2}{8\sigma}\left(\frac{y}{L} - \frac{y^2}{L^2}\right). \quad (39)$$

There appears to be considerable difficulty in evaluating equation (38), since there will always be an optically non-thick region in the band wings. A procedure employed in [9] to evaluate κ_R consists of assuming an Elsasser model, which replaces the band by equally spaced lines of equal intensity. However, this is simply an extension of the box model, and in fact reduces to the box model for $\beta \to \infty$. Since no account is made of the wing regions of the band, and since the wing regions are of prime importance for large path lengths, this procedure is open to some question.

COMPARISON OF RESULTS

For the sake of brevity, it will be sufficient for comparative purposes to compare centerline temperatures as predicted by the various spectral models; that is, $T_c = T(y = L/2)$. Recall that both the box model and the band absorptance model describe the temperature profile in terms of the quantity φ defined by equation (35). It will thus be convenient to describe the centerline temperature, in terms of Planck's function evaluated at the band center, by the dimensionless group

$$\varphi\left(\frac{u_0}{2}\right) = \frac{e_{\omega c}(T_c) - e_{\omega_c}(T_1)}{Q/PS(T_1)}. \quad (40)$$

The box model parameter $\bar{\tau}_0$ and the band absorptance parameter u_0 are related for carbon monoxide through equation (33). Results for $\varphi(u_0/2)$, as obtained from equation (29) for the box model and from the solution of equation (36) for the band absorptance model, are illustrated in Fig. 3. Furthermore, the optically

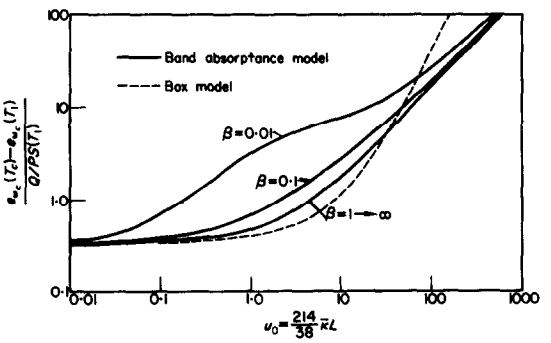

FIG. 3. Comparison of results for carbon monoxide.

thin limit follows from equation (30) to be

$$\varphi\left(\frac{u_0}{2}\right) = \frac{1}{3}. \tag{41}$$

Note that the band absorptance model approaches the correct optically thin limit, with the results becoming independent of the line structure parameter in this limit. This of course is consistent with the fact that the line structure of the band has no effect under optically thin conditions, with the radiative transfer process depending solely upon the area under the κ_ω/P vs. wave number curve. As the path length u_0 is increased, the band absorptance results show a greater departure from the optically thin limit for small values of the line structure parameter β. This is consistent with previous discussion. For small values of β (low pressures), the maximum value of κ_ω/P will exceed that for higher pressures, and correspondingly, for increasing u_0, optically nonthin conditions will first occur for small β values.

The maximum influence of line structure exists for intermediate values of u_0, while for large values of path length the band absorptance results again become insensitive to β. This corresponds to the path length range for which the total band absorptance, as given by equation (14), reduces to the logarithmic asymptote $\bar{A} = \ln u$, and is thus independent of β. As previously discussed, the physical implication is that the central portion of the band is opaque, with radiative transfer occurring solely within the wing regions. Edwards *et al.* [14] have pointed out that the wings possess a more continuous structure than the central portion of the band, with the consequent supression of the importance of line structure at large path lengths.

With reference to the box model results, recall that the box model neglects line structure and thus is a large β approximation. From Fig. 3 it is seen that the box model solution departs from the optically thin solution more slowly than its band absorptance counterpart ($\beta = 1 \to \infty$). This is easily explained on physical grounds. In the central portion of the band the box model underpredicts the value of the spectral absorption coefficient (see Fig. 2) such that the box model will yield optically thin results for greater values of u_0 than will the band absorptance model. At large values of u_0 the box model overpredicts the centerline temperature due to the neglect of the band wings. For large path lengths the wing regions contribute primarily to radiative transfer. Since the box model neglects the wings, it underestimates the ability of the gas to transfer radiant energy for large u_0 values, and consequently overpredicts the centerline temperature.

In regard to comparing the gray gas and optically thick solutions with the band absorptance solution, note that equations (21) and (39) describe the centerline temperature as T_c^4, while T_c is given in terms of Planck's function for the band absorptance model. To express T_c in a comparable form, it should be recalled that the present analyses are based on the assumption that κ_ω is independent of temperature; that is, restriction has been made to small temperature differences. In line with

this, the linearizations

$$T^4 - T_1^4 \simeq 4T_1^3(T - T_1)$$

and

$$e_{\omega_c}(T) - e_{\omega_c}(T_1) \simeq \left(\frac{de_{\omega_c}}{dT}\right)_{T=T_1}(T - T_1)$$

are applicable. On eliminating $(T - T_1)$ from these two expressions, there is obtained

$$T^4 - T_1^4 \simeq \frac{4T_1^3}{(de_{\omega_c}/dT)_{T=T1}}[e_{\omega_c}(T) - e_{\omega_c}(T_1)]. \quad (42)$$

Through use of equation (42), it is possible to express $(T_c^4 - T_1^4)$ in terms of $\varphi(u_0/2)$. Specifically, from equation (35) this yields

$$T_c^4 - T_1^4 \simeq \frac{4T_1^3 Q}{PS(T_1)(de_{\omega_c}/dT)_{T=T_1}}\varphi\left(\frac{u_0}{2}\right) \quad (43)$$

such that equations (21) and (39) may be rephrased to yield results for $\varphi(u_0/2)$. Note that this necessitates specifying not only the gas but also the wall temperature T_1. The band absorptance, gray gas, and optically thick solutions are compared in Figs. 4 and 5 for $T_1 = 1000°K$ and $2000°K$ respectively.

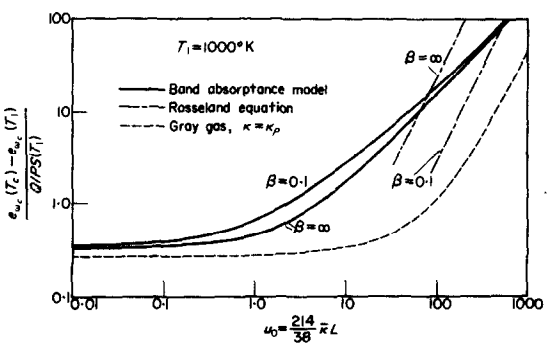

FIG. 4. Comparison of results for carbon monoxide, $T_1 = 1000°K$.

It is seen that the gray gas solution is in poor agreement with the more realistic band absorptance solution, even in the optically thin limit. This disagreement in the thin limit is a consequence of two errors which are inherent in the gray gas analysis. The first is that the gray gas absorption coefficient has been assumed to be the Planck mean. This is correct only with regard to emission, whereas in the thin limit the correct mean absorption coefficient is the modified Planck mean. Secondly, in the gray gas solution it has been assumed that κ_P is independent of temperature, and as previously discussed this is incorrect even for small temperature differences.

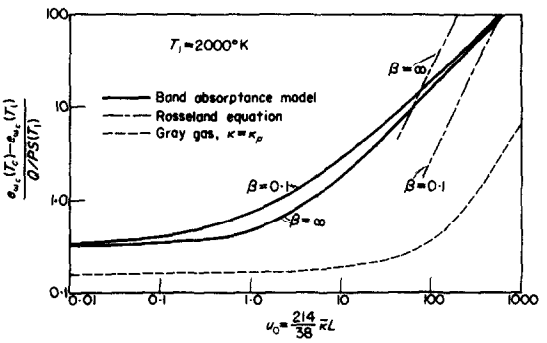

FIG. 5. Comparison of results for carbon monoxide, $T_1 = 2000°K$.

Also illustrated in Figs. 4 and 5 are results based upon the optically thick (or Rosseland) equation applied to a single band. Values of the Rosseland coefficient κ_R for the CO fundamental band were taken from [9], and the method of calculating this coefficient, employing the Elsasser model, was described previously.

Presumably, the optically thick solution should describe the asymptotic behavior for large values of the path length u_0. Note from Figs. 4 and 5, however, that the Rosseland equation curves have a considerably different slope than appears to be asymptotically achieved by the band absorptance model. The slope of the Rosseland curves actually coincides with that approached by the box model and gray gas solutions. Furthermore, the Rosseland results show a strong dependence upon the line structure parameter β and this is not in accord with previous conclusions.

These apparent shortcomings of the optically thick solution are evidently due to the fact that vibration–rotation bands can never be

optically thick over the entire spectral range where radiative transfer is important. For example, consider the limit $\beta \to \infty$, such that the variation κ_ω/P with wave number is the line-average value illustrated in Fig. 2. In the band wings, κ_ω/P approaches zero asymptotically, so that for large values of u_0 the wings will constitute regions for which there will be a continuous transition from the opaque to the transparent limits. These wing regions will, of course, be the sole wave number regions for which radiative transfer within the gas occurs, since the central portion of the band will be opaque. In other words, for large u_0 optically nonthick radiation will occur in the band wings, whereas in [9] it was necessary to neglect the wing regions in order to define a band Rosseland coefficient.*

A final comment regarding the limit of large u_0 concerns the neglect of the first overtone band. This band, which has been neglected in the present work, will become important for large path lengths. Its inclusion, however, would not alter the conclusion concerning the applicability of the Rosseland equation to vibration–rotation bands.

* Any attempt to include the wing regions will result in a diverging integral in equation (38).

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Résumé—L'objet de cette recherche est d'étudier plusieurs méthodes approchées d'analyse du transport de chaleur par rayonnement infrarouge dans des gaz non gris et nonisothermes. Dans ce but, on a choisi un système physique très simple sonsistant en un gaz compris entre deux plaques noires parallèles infinies ayant la même température uniforme. Il existe une source (ou un puits) de chaleur uniforme dans le gaz. De plus, on se restreint aux gaz ayant une seule bande fondamentale de vibration–rotation, c'est-à-dire, aux gaz diatomiques.

On trouve que pour des épaisseurs optiques intermédiaires, la structure des lignes de la bande de vibration–rotation, peut avoir un effet important sur la distribution de température dans le gaz. Des prévisions basées sur l'hypothèse du gaz gris conduisent à des grandes erreurs. On montre plus loin que l'on ne, peut pas appliquer le cas limite optiquement épais à toute une bande de vibration–rotation, puisqu'il y aura toujours une région non épaisse optiquement dans les extrémités de la bande, et que de telles régions contribueront d'une façon importante au processus de transport par rayonnement.

Zusammenfassung—Es werden verschiedene Näherungsverfahren zur Berechnung des Wärmeübergangs durch Strahlung in nicht-grauen und nicht isothermen Gasen untersucht. Ein sehr einfaches physikalisches

System, nämlich zwei parallele, unendlich ausgedehnte schwarze Platten gleicher Temperatur, zwischen denen sich das Gas befindet, wird zu Grunde gelegt. Im Gas befindet sich eine homogene Wärmequelle (oder Senke). Weiterhin werden die Überlegungen beschränkt auf Gase mit einer einzigen Grundschwingungs–Rotationsbande, d.h. auf zweiatomige Gase.

Es ergibt sich, dass die Linienstruktur der Schwingungs–Rotationsbanden bei mittleren optischen Dicken einen beträchtlichen Einfluss auf die Temperaturverteilung im Gas haben kann. Es wird gezeigt, dass Vorhersagen, die auf der annahme eines grauen Gases beruhen, sehr fehlerhaft sind. Ausserdem wird gezeigt, dass man die Näherung für den optisch dicken Fall nicht für die gesamte Schwingungs–Rotationsbande verwenden kann, weil in der Bande immer optisch nicht-dicke Bereiche sein werden, die erheblich zum Wärmeübergang durch Strahlung beitragen.

Аннотация—Предметом данного исследования является изучение различных приближенных методов анализа инфракрасного лучистого теплообмена в неизотермических средах несерых газов. С этой целью была выбрана очень простая физическая система, состоящая из газа, заключенного между двумя бесконечными параллельными черными изотермическими пластинками с одинаковой температурой. В газовой среде имеется однородный источник (или сток) тепла. Исследовались только газы, имеющие одну основную вибрационно-вращательную полосу поглощения, т.е. двухатомные газы.

Установлено, что для промежуточных значений оптической толщины линейная структура колебательно-вращательной полосы поглощения может значительно влиять на распределение температуры в газе. Показано, что расчеты, основанные на допущениях о сером газе, дают большие ошибки. Далее на примерах показано, что предельно большие значения оптической толщины нельзя применить ко всей вибрационно-вращательной полосе, т.к. на концах полосы всегда существуют области с небольшими значениями оптической толщины, которые имеют большое значение в процессе лучистого теплообмена.